ON THE ZEROS OF FUNCTIONS DEFINED BY HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS CONTAINING A PARAMETER

BY

MERLIN GRANT SMITH

A. B. Greenville College, 1915.

A. M. University of Illinois, 1916.

THESIS

Submitted in Partial Fulfillment of the Requirements for the

Degree of

DOCTOR OF PHILOSOPHY

IN MATHEMATICS

IN

THE GRADUATE SCHOOL

OF THE

UNIVERSITY OF ILLINOIS

1918

UNIVERSITY OF ILLINOIS

THE GRADUATE SCHOOL

191

I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Merlin Grant Smith

ENTITLED On the Zeros of Functions Defined by Homogeneous Linear Differential Equations Containing a Parameter.

BE ACCEPTED AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE OF Doctor of Philosophy

R. D. Carmichael
In Charge of Thesis

E. J. Townsend
Head of Department

Recommendation concurred in:*

Aubrey J. Kempner
C. H. Sisam
F. R. Watson
Joel Stebbins

Committee on Final Examination*

*Required for doctor's degree but not for master's.

TABLE OF CONTENTS

ON THE ZEROS OF FUNCTIONS DEFINED BY HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS CONTAINING A PARAMETER

Introduction.

In his Fifth International Congress paper Bôcher* has pointed out the desirability of obtaining theorems for differential equations of order higher than two which would generalize the theorems derived by Sturm† for second order equations. Certain generalizations in this direction have been made by Birkhoff‡ and Davidglou§ for self-adjoint equations of order three and four respectively. Carmichael**, in a more recent memoir, has obtained some general theorems of comparison for equations of any order n. The purpose of the present paper is to treat the oscillation problem suggested at the close of the last article of Carmichael's paper and to show how the theorems obtained are in the nature of generalizations of certain of the Sturmian results.

After stating in §I certain well-known formulae for a differential equation and its adjoint and an immediate consequence of these, I give in §II

....

* M.Bôcher, Proceedings of the Fifth International Congress of Mathematicians, vol.I (1912), pp. 163-195.

† C.Sturm, Journal de Mathématiques, vol.1 (1836), pp.106-186. See also M.Bôcher, Bulletin of the American Mathematical Society, ser.2, vol.4 (1898), pp.295-313, 365-376; also, Leçons sur les Méthodes de Sturm, Paris, 1917.

‡ G.D.Birkhoff, Annals of Mathematics, vol.12 (1911), pp.103-127.

§ A.Davidglou, Annales de l'École Normale Supérieure, ser.3, vol.17 (1900), pp.359-444; also, ser.3, vol.22 (1905), pp.539-565.

**R.D.Carmichael, Annals of Mathematics, vol.19 (1918), pp.159-171.

an extension of a theorem, in the paper by Carmichael, upon which the remainder of this paper is based. §III contains an outline of the method to be employed and suggests the possibility of obtaining a variety of theorems of oscillation for a single differential equation. In this section I also carry out the details by means of a particular auxiliary equation and derive the First General Theorem of Oscillation. Some difficulties arise in applying this theorem to a particular equation; and, in §IV, I introduce an equation of special form for which the theorem mentioned above becomes particularly elegant. In each of the following two sections I derive two theorems, analogous to the two just mentioned, by using particular solutions of other suitable auxiliary differential equations. The next and last section is devoted to an application of the preceeding results to equations not involving a parameter. I point out that these theorems form a generalization of certain of the classic Sturmian results; but the general oscillation problem involved has not been solved, for the theorems obtained state only sufficient conditions that solutions of a given equation oscillate in a fixed interval.

I. Preliminary Formulae.

We will consider the homogeneous linear parametric differential equation

$$L(u) \equiv l_n(x,\lambda)\frac{d^n u}{dx^n} + \lambda l_{n-1}(x,\lambda)\frac{d^{n-1}u}{dx^{n-1}} + \cdots + \lambda^{n-1} l_1(x,\lambda)\frac{du}{dx} + \lambda^n l_0(x,\lambda)u = 0, \quad (1)$$

in which we assume: that for λ fixed and greater than or equal to a given positive quantity $\bar{\lambda}$ the functions $l_n, l_{n-1}, \cdots, l_1, l_0$ are real-valued, single-valued and continuous functions of x for x in a given (finite) interval defined by the inequalities $a \leq x \leq b$, and that l_n is of one sign, say positive, in (a,b); that the derivatives* of l_i, up to order i inclusive, $i = 1, 2, \cdots, n$, are likewise real-valued, ~~and~~ single-valued and continuous functions of x for the fixed value of λ; that for a fixed value of x in (a,b), the functions l_i and their derivatives up to order i inclusive, $i = 0, 1, \cdots, n$, are real-valued single-valued and continuous functions of λ in the range $\lambda \geq \bar{\lambda}$.

The Lagrange adjoint equation associated with (1) is

$$M(v) \equiv (-1)^n \frac{d^n}{dx^n}(l_n v) + (-1)^{n-1}\lambda\frac{d^{n-1}}{dx^{n-1}}(l_{n-1}v) + \cdots - \lambda^{n-1}\frac{d}{dx}(l_1 v) + \lambda^n l_0 v = 0,$$

This, when arranged according to the derivatives of v, may be written thus

$$M(v) \equiv m_n(x,\lambda)\frac{d^n v}{dx^n} + \lambda m_{n-1}(x,\lambda)\frac{d^{n-1}v}{dx^{n-1}} + \cdots + \lambda^{n-1} m_1(x,\lambda)\frac{dv}{dx} + \lambda^n m_0(x,\lambda)v = 0, \quad (2)$$

where the coefficients are defined by the following relations:

....

* The derivatives will always be taken with respect to x; ~~and~~ sometimes they will be denoted by primes.

$$
\begin{aligned}
(-1)^n m_n &= l_n \,, \\
(-1)^{n-1} m_{n-1} &= l_{n-1} - n\frac{l_n'}{\lambda} \,, \\
(-1)^{n-2} m_{n-2} &= l_{n-2} - (n-1)\frac{l_{n-1}'}{\lambda} + \frac{n(n-1)}{|\underline{2}}\,\frac{l_n''}{\lambda^2} \,, \\
(-1)^{n-3} m_{n-3} &= l_{n-3} - (n-2)\frac{l_{n-2}'}{\lambda} + \frac{(n-1)(n-2)}{|\underline{2}}\,\frac{l_{n-1}''}{\lambda^2} - \frac{n(n-1)(n-2)}{|\underline{3}}\,\frac{l_n'''}{\lambda^3} \,,
\end{aligned}
\tag{3}
$$

....

$$
m_0 = l_0 - \frac{l_1'}{\lambda} + \frac{l_2''}{\lambda^2} - \cdots + (-1)^n \frac{l_n^{(n)}}{\lambda^n} \,.
$$

It is easy to establish the well-known Lagrange identity*,

$$
vL(u) - uM(v) \equiv \frac{d}{dx} R(u,v) \,, \tag{4}
$$

where

$$
\begin{aligned}
R(u,v) \equiv {} & \left[l_n v\right]\frac{d^{n-1}u}{dx^{n-1}} + \left[\lambda l_{n-1} v - \frac{d}{dx}(l_n v)\right]\frac{d^{n-2}u}{dx^{n-2}} \\
& + \left[\lambda^2 l_{n-2} v - \lambda\frac{d}{dx}(l_{n-1} v) + \frac{d^2}{dx^2}(l_n v)\right]\frac{d^{n-3}u}{dx^{n-3}} + \cdots \\
& + \left[\lambda^{n-1} l_1 v - \lambda^{n-2}\frac{d}{dx}(l_2 v) + \lambda^{n-3}\frac{d^2}{dx^2}(l_3 v) - \cdots + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(l_n v)\right] u.
\end{aligned}
$$

We also introduce the forms

$$
P(y) \equiv p_n(x,\lambda)\frac{d^n v}{dx^n} + \lambda p_{n-1}(x,\lambda)\frac{d^{n-1}v}{dx^{n-1}} + \cdots + \lambda^{n-1} p_1(x,\lambda)\frac{dv}{dx} + \lambda^n p_0(x,\lambda) v
$$

and

$$
D(v) \equiv P(y) - M(v). \tag{5}
$$

Here we assume: that, when λ has a fixed value as designated above, p_n, $p_{n-1}/p_n, \cdots, p_n/p_n, p_0/p_n$ are continuous, real-valued and single-valued functions of x, in (a,b), except possibly for singularities of a special character mentioned

....

* See, for instance, Forsyth's Theory of Differential Equations, Vol. IV, p.252.

below; that, when x has a fixed value in (a,b) $p_n, p_{n-1}/p_n, \cdots, p_1/p_n, p_0/p_n$ are continuous, real-valued and single-valued functions of λ, except for the singularities just referred to.

In the following treatment $u_\kappa(x,\lambda)$, $v_\kappa(x,\lambda)$, $y_\kappa(x,\lambda)$ will denote particular solutions of the equations $L(u) = 0$, $M(v) = 0$ and $P(y) = 0$ respectively. The possible singularities to be admitted for the functions* p_n, $p_{n-1}/p_n, \cdots$, p_1/p_n, p_0/p_n are such that solutions y_κ of $P(y) = 0$ exist so that $D(y_\kappa)$ is continuous for x in (a,b) and λ fixed and greater than or equal to $\bar{\lambda}$. By $y_\kappa(x,\lambda)$ we denote only those solutions of $P(y) = 0$ for which $D(y_\kappa)$ has this continuity property.

As a special case let $L(u_\lambda) = 0$ and $P(v_t) = 0$; evidently $v_t L(u_\lambda) - u_\lambda P(v_t) = 0$. Then from (5) and (4)

$$\frac{d}{dx}\left\{R(u_\lambda, v_t)\right\} = u_\lambda D(v_t).$$

If α and β are two points in (a,b), the conditions on the functions in this equation allow us to integrate both sides from α to β and hence we have,

$$R(u_\lambda, v_t)\Big|_{x=\alpha}^{x=\beta} = \int_\alpha^\beta u_\lambda D(v_t)\, dx. \qquad (6)$$

* We should note here that p_i/p_n may have poles of as high order as $n-i-1$, $i = 0, 1, 2, \cdots, n-1$, at those points where v_κ has zeros of order $n-1$, without disturbing the continuity of $D(v_\kappa)$.

II. General Theorem of Comparison.

From the formulae of the previous section we can prove the following theorem, which is an extension of a theorem due to Carmichael.*

Theorem. *Let y_κ be a solution of the equation $P(y) = 0$, and, for a fixed value of λ in the range $\lambda \geq \bar{\lambda}$, suppose that it vanishes to the $(n-1)^{st}$ order at α and at β and is positive*† *between α and β while $D(y_\kappa)$ is not identically zero in (α,β) and is not positive at any point in (α,β). Let $u(x,\lambda)$ be any solution of the equation $L(u) = 0$.*

Then:

1) *$u(x,\lambda)$ vanishes in the interior of (α,β) when n is even;*

2) *$u(x,\lambda)$ vanishes in the interior of (α,β) when n is odd provided in this case that $u(x,\lambda)$ is positive at some point in (α,β) and that*

$$l_n(\beta,\lambda)y_\kappa^{(n-1)}(\beta,\lambda)u(\beta,\lambda) \geq l_n(\alpha,\lambda)y_\kappa^{(n-1)}(\alpha,\lambda)u(\alpha,\lambda). \tag{7}$$

This theorem is proved by the indirect method from the equations of §1, where we assume the parameter λ fixed and greater than or equal to $\bar{\lambda}$. By hypothesis: l_n is positive throughout the interval (α,β) which is included in (a,b); $D(y_\kappa)$ is negative or zero (but not identically zero) in (α,β); y_κ is positive between α and β; in the neighborhood of α to the right, all the first $n-1$ derivatives of y_κ are positive and hence $y_\kappa^{(n-1)}$ is positive at α; in the neighborhood of β to the left, when considering the first $n-1$ derivatives of y_κ, the odd ordered derivatives are negative while the even ordered are positive, hence $y_\kappa^{(n-1)}$ is negative at β when n is even and positive when n is odd. Moreover, let us suppose that $u(x,\lambda)$ is of one sign,

....

* Loc. cit., p. 170.

† In case y_κ is negative between α and β, the condition to be imposed upon $D(y_\kappa)$ is that it is not negative throughout (α,β).

say positive, throughout (α,β).

From equation (6) we have,

$$(-1)^{n-1} l_n(x,\lambda) y_k^{(n-1)}(x,\lambda) u(x,\lambda) \Big|_{x=\alpha}^{x=\beta} = \int_\alpha^\beta u\, D(y_k)\, dx. \qquad (8)$$

From equation (8), if n is even, we have

$$l_n(\alpha,\lambda) y_k^{(n-1)}(\alpha,\lambda) u(\alpha,\lambda) - l_n(\beta,\lambda) y_k^{(n-1)}(\beta,\lambda) u(\beta,\lambda) = \int_\alpha^\beta u\, D(y_k)\, dx.$$

The signs of the quantities $l_n(\alpha,\lambda)$, $y_k^{(n-1)}(\alpha,\lambda)$, $u(\alpha,\lambda)$, $l_n(\beta,\lambda)$, and $u(\beta,\lambda)$ are all positive while $y_k^{(n-1)}(\beta,\lambda)$ is negative; hence the left hand member is positive while the right is negative. Thus, when n is even, the assumption that $u(x,\lambda)$ is of one sign throughout (α,β) leads to a contradiction and therefore $u(x,\lambda)$ vanishes at least once in the interior of the interval.

When n is odd, equation (8) evidently is

$$l_n(\beta,\lambda) y_k^{(n-1)}(\beta,\lambda) u(\beta,\lambda) - l_n(\alpha,\lambda) y_k^{(n-1)}(\alpha,\lambda) u(\alpha,\lambda) = \int_\alpha^\beta u\, D(y_k)\, dx.$$

Since the second member is negative the last equation is contradictory with relation (7). Hence, when n is odd, the assumption thaf $u(x,\lambda)$ is positive throughout (α,β) is untenable.

This completes the proof of the theorem.

III. First General Theorem of Oscillation.

From the considerations of the two preceding sections, we are now able to derive certain theorems concerning the zeros of any solution of the differential equation $L(u) = 0$. The results which are to follow are based primarily upon the theorem stated above. By an analysis of this theorem we find that all its hypotheses, except the one on $D(v_k)$, are easily satisfied if we choose the auxiliary differential equation $P(y) = 0$ in such a way that it has the $(n-1)^{st}$ power of a suitable periodic function as a particular solution, since for λ and an arbitrary constant of integration appropriately chosen this solution has consecutive zeros at α and β, two points in the interval (a,b). In order to simplify the form of $D(v_k)$, we will choose $p_n, p_{n-1}, \cdots, p_2, p_1$ identical respectively with $m_n, m_{n-1}, \cdots, m_2, m_1$; then

$$D(v_k) \equiv \lambda^n (p_0 - m_0)\ v_k\ .$$

Furthermore if we substitute the chosen particular solution y_k for y in $P(y) = 0$ we can solve for p_0, the only undetermined coefficient still remaining in $P(y)$, in terms of $m_n, m_{n-1}, \cdots, m_2, m_1$, and, by means of equations (3), in terms of the first n coefficients of $L(u)$ and their derivatives; since by (3) m_0 is also expressible in terms of the coefficients of $L(u)$ or their derivatives, the condition to be imposed upon $D(v_k)$ involves only the coefficients of the given equation $L(u) = 0$ and a known chosen function. Besides, if

$$p_0 - m_0 \leq 0, \quad p_0 - m_0 \not\equiv 0,$$

the hypothesis on $D(v_k)$ in the above theorem is fulfilled. Solving this condition for $l_0(x,\lambda)$, the only coefficient of $L(u)$ whose derivative is not

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

involved in the inequality, we obtain an inequality which will form one of the fundamental hypotheses of the new theorems which we derive.

In the first case we select, as a solution of $P(y) = 0$, the function $y_1 = \cos^{n-1}\lambda(x-c)$ which, as λ increases, has an increasing number of $(n-1)^{st}$ order zeros in the interval $a \leq x \leq b$. We can write

$$
\begin{aligned}
y_1 &= \cos^{n-1}\lambda(x-c), \\
y_1' &= \left[-(n-1)\tan\lambda(x-c)\right]\lambda y_1, \\
y_1'' &= \left[(n-1)(n-2)\tan^2\lambda(x-c) - (n-1)\right]\lambda^2 y_1, \qquad (9)\\
y_1''' &= \left[-(n-1)(n-2)(n-3)\tan^3\lambda(x-c) + (n-1)(3n-5)\tan\lambda(x-c)\right]\lambda^3 y_1, \\
y_1'''' &= \left[(n-1)\cdots(n-4)\tan^4\lambda(x-c)-(n-1)(n-2)(6n-14)\tan^2\lambda(x-c)+(n-1)(3n-5)\right]\lambda^4 y_1,
\end{aligned}
$$

....

To facilitate the computation let us place

$$y_1^{(r)} = w_r y_1;$$

from this the partial difference and differential recurrence relation

$$w_{r+1} = w_r' - (n-1)\lambda\tan\lambda(x-c)\cdot w_r \qquad (10)$$

is obtained, of which r is the recurrence variable. If the sum of the first $n+1$ equations of (9), after multiplication in order by $\lambda^n p_0$, $\lambda^{n-1}p_1, \cdots,$ λp_{n-1}, p_n is taken, the first member is $P(y_1)$; and this has the value zero. If the resulting equation is solved for $\lambda^n p_0$ we have

$$-\lambda^n p_0 = w_1\lambda^{n-1}p_1 + w_2\lambda^{n-2}p_2 + \cdots + w_{n-1}\lambda p_{n-1} + w_n p_n,$$

where the functions $w_1 = -(n-1)\lambda\tan\lambda(x-c)$, $w_2, \cdots, w_{n-1}$, w_n are determined by means of (10) and are linear functions of the even or odd powers of $\tan\lambda(x-c)$ according as the subscript of w_r, $r = 1, 2, \cdots, n$, is even or odd;

moreover, w_λ has as a factor λ^λ. The highest power of $\tan\lambda(x-c)$ in w_λ is r, except for r = n, in which case the highest power is n-2. According to the statement above we now specify that the p's in this value of $-\lambda^n p_0$ shall be identical respectively with the coefficients of M(v) having the same subscripts. Dividing the equation through by $-\lambda^n$ and arranging the value of p_0 according to the powers of $\tan\lambda(x-c)$, we may write

$$p_0 = A_0 + A_1 \tan\lambda(x-c) + \cdots + A_{n-1}\tan^{n-1}\lambda(x-c),$$

where the functions A_i, $i = 0, 1, 2, \cdots, n-1$, are linear in the functions p_k, $k = 1, 2, \cdots, n$, and contain only those functions for which k is even or odd according as i is even or odd respectively and for which $k \geq i$. Moreover, the coefficient of p_i in A_i is a constant different from zero. Now, expressing the p's with subscripts $1, 2, \cdots, n$, by means of (3), in terms of the coefficients of L(u) we find that the A's are determinate linear functions of the coefficients of the given equation and their derivatives and hence that p_0 is a determinate function.

In a later discussion it will be useful to know the explicit form of certain of the A's; hence we include here the following formulae, given in terms of a general order n. These formulae are sufficient for equations of order as high as six, since the coefficient of p_{i+1} in A_0 is always the same as the coefficient of p_i in A_1.

$$\begin{aligned}
A_{n-1} &= (-1)^n \lfloor n-1 \; p_{n-1}, \\
A_{n-2} &= (-1)^n \frac{\lfloor n-1}{\lfloor 1}\left[S'_{n-1}p_n - p_{n-2}\right], \\
A_{n-3} &= (-1)^n \frac{\lfloor n-1}{\lfloor 2}\left[-S'_{n-2}p_{n-1} + p_{n-3}\right], \\
A_{n-4} &= (-1)^n \frac{\lfloor n-1}{\lfloor 3}\left[-S''_{n-3}p_n + S'_{n-3}p_{n-2} - p_{n-4}\right],
\end{aligned} \tag{11}$$

$$A_{n-5} = (-1)^n \frac{\lfloor n-1}{\lfloor 4} \left[S''_{n-4} p_{n-1} - S'_{n-4} p_{n-3} + p_{n-5} \right],$$

$$\cdots\cdots\cdots\cdots\cdots\cdots ,$$

where the quantities S_k^i may be defined as follows: If

$$k^{(s)} = k(k-1)(k-2)\cdots(k-s+1),$$

then

$$S'_k = (n-1) + 2(n-2) + \cdots + k(n-k)$$

$$= -\frac{k^{(3)}}{3} + \frac{n-3}{2} k^{(2)} + (n-1)k^{(1)};$$

$$S''_k = (n-1)S'_2 + 2(n-2)S'_3 + \cdots + k(n-k)S'_{k+1}$$

$$= \frac{1}{3\cdot 6} k^{(6)} - \frac{5n-39}{5\cdot 6} k^{(5)} + \frac{3n^2-68n+233}{4\cdot 6} k^{(4)} + \frac{9n^2-84n+163}{3\cdot 2} k^{(3)}$$

$$+ \frac{9n^2-44n+51}{2} k^{(2)} + (3n^2-8n+5)\, k^{(1)}.$$

For further convenience, when λ is fixed and greater than or equal to $\bar{\lambda}$, let us denote the consecutive zeros of y_k, where the arbitrary constant is c chosen so that y_k vanishes at the point a of the interval (a,b), by $\mu_0 = a, \mu_1, \mu_2, \cdots, \mu_s$ such that $\mu_0 < \mu_1 < \mu_2 < \cdots < \mu_s \leq b$. The value of c for the case of y_1 is $a - \pi/2\lambda$. It is readily seen that y_k has s+1 zeros of order n-1 in (a,b) if $\lambda \geq s\pi/(b-a)$.

By the theorem of the preceding section, if $D(y_1)$ is non-positive or non-negative but not identically zero in an interval within which y_1 is positive or negative respectively and has zeros of order n-1 at the ends of that interval, then any solution $u(x,\lambda)$ of $L(u) = 0$ vanishes in the interior of this interval when n is even. For this case, $\cos^{n-1}\lambda(x-c)$ is alter-

nately positive and negative in consecutive intervals of this sort and if the theorem is to be applicable for a range (a,b) which includes more than two consecutive zeros of y_1, $D(y_1)$ must change sign with y_1 and hence from the relation $D(y_1) = \lambda^n(p_0 - m_0)y_1$ we must have $p_0 - m_0 \leq 0$ throughout the interval (a,b).

If n is odd $\cos^{n-1}\lambda(x-c)$ is always positive or zero. According to the theorem just referred to, if $u(x,\lambda)$ is positive at some point in each of the intervals $(\mu_0,\mu_1),(\mu_1,\mu_2),\cdots(\mu_{s-1},\mu_s)$, if $p_0 - m_0 \leq 0$ and if

$$l_n(\mu_0,\lambda)v_1^{(n-1)}(\mu_0,\lambda)u(\mu_0,\lambda) \leq l_n(\mu_1,\lambda)v_1^{(n-1)}(\mu_1,\lambda)u(\mu_1,\lambda) \leq$$

$$\cdots \leq l_n(\mu_s,\lambda)v_1^{(n-1)}(\mu_s,\lambda)u(\mu_s,\lambda),$$

then $u(x,\lambda)$ vanishes at least once in each of the sub-intervals of (a,b) of length π/λ measuring from the point a.

The condition $p_0 - m_0 \leq 0$ yields an inequality governing the coefficients of the equation $L(u) = 0$. Employing this, the results obtained may be stated in the form of the following fundamental theorem.

Theorem I. *If the parameter λ, in the equation $L(u) = 0$, is fixed and satisfies the relations*,

$$\lambda \geq \bar{\lambda} \quad , \lambda \geq \frac{s\pi}{b-a} \;,$$

and if for every x in the interval $a \leq x \leq b$

$$l_0(x,\lambda) \geq A_0 + A_1\tan\lambda(x-c) + \cdots + A_{n-1}\tan^{n-1}\lambda(x-c)$$

$$+ \frac{l_1'}{\lambda} - \frac{l_2''}{\lambda^2} + \cdots + (-1)^{n+1}\frac{l_n^{(n)}}{\lambda^n} \;,\; c = a - \pi/2\lambda, \qquad (12)$$

the sign of equality not holding throughout (a,b), *then any solution* $u(x,\lambda)$ *of* $L(u) = 0$ *vanishes at least once in the interior of each sub-interval of*

[illegible]

(a,b) *of length* π/λ *measuring from the point* a *and hence has at least* s *zeros in* (a,b):

1) *when* n *is even;*

2) *when* n *is odd, provided in this case that* $u(x,\lambda)$ *is positive at some point in each of the intervals* $(\mu_0,\mu_1),(\mu_1,\mu_2),\cdots,(\mu_{s-1},\mu_s)$ *and that*

$$l_n(\mu_0,\lambda)v_1^{(n-1)}(\mu_0,\lambda)u(\mu_0,\lambda) \leq l_n(\mu_1,\lambda)v_1^{(n-1)}(\mu_1,\lambda)u(\mu_1,\lambda) \leq$$
$$\cdots \leq l_n(\mu_s,\lambda)v_1^{(n-1)}(\mu_s,\lambda)u(\mu_s,\lambda),$$

where $\mu_0 = a,\ \mu_1,\ \cdots,\mu_s$ *are the consecutive zeros of* v_1 *in* (a,b).

IV. Introduction of a Special Form of the Equation.

The theorem of the preceding section is more readily applicable to equations of even order than to those of odd order, since, for the case of n even, we can be assured that the function $u(x,\lambda)$ vanishes at least s times in the interval (a,b) without any hypotheses on the properties of this function other than that it satisfies the equation $L(u) = 0$, while, for n odd, we know that $u(x,\lambda)$ vanishes in the interval only when, in addition to satisfying the equation $L(u) = 0$, it also meets certain hypotheses as to its specific character at a given number of points of this range.

In what follows we will confine ourselves to the study of equations of even order and moreover shall seek to specialize the coefficients of $L(u)$ so that the hypothesis on l_0 takes a form which makes the theorem more easy to apply. Such a specialization can be made by requiring that, in the inequality (12), the coefficients of the odd powers of $\tan\lambda(x-c)$ shall be such that they each are expressible as the product of $\tan\lambda(x-c)$ and a function which is non-positive throughout the interval (a,b). It is convenient for us to make the further restriction that this non-positive factor is zero and hence that the coefficients A_i, $i = 1,3,5,\cdots$, are identically zero in (a,b). From the definition, in equations (11), of the functions A_i, $i = 1,3,5,\cdots$, for the case of n even, we see that the vanishing of these functions necessitates the vanishing of p_i, $i = 1,3,5,\cdots$, and conversely; hence the adjoint of $L(u) = 0$, in the present case, must be an equation in which the coefficients of the odd ordered derivatives are identically zero.

To determine what form $L(u)$ may take so that the adjoint equation satisfies the above restrictions we note that the equation adjoint to the adjoint of a given equation is the given equation. Hence the adjoint of

$$M(v) \equiv (-1)^n \frac{d^n}{dx^n}(l_n v) + (-1)^{n-1}\lambda \frac{d^{n-1}}{dx^{n-1}}(l_{n-1}v) + \cdots - \lambda^{n-1}\frac{d}{dx}(l_1 v) + \lambda^n l_0 v = 0$$

is $L(u) = 0$ as written in (1). $L(u) = 0$ may then be written in the form

$$L(u) \equiv (-1)^n \frac{d^n}{dx^n}(m_n u) + (-1)^{n-1}\lambda \frac{d^{n-1}}{dx^{n-1}}(m_{n-1}u) + \cdots - \lambda^{n-1}\frac{d}{dx}(m_1 u) + \lambda^n m_0 u = 0,$$

for its adjoint is $M(v) = 0$ as expressed in (2). Since $L(u) = 0$ is unrestricted, any differential equation can be reduced to the form of the last equation.

Since we wish to confine our attention to an equation of even order in which the coefficients, corresponding to the m's with odd subscripts in the last equation above, are zero, we will introduce the notation

$$E(u) \equiv \frac{d^{2m}}{dx^{2m}}\left[e_{2m}(x,\lambda)u\right] + \lambda^2 \frac{d^{2m-2}}{dx^{2m-2}}\left[e_{2m-2}(x,\lambda)u\right] + \cdots$$

$$+ \lambda^{2m-2}\frac{d^2}{dx^2}\left[e_2(x,\lambda)u\right] + \lambda^{2m} e_0(x,\lambda)u = 0, \tag{13}$$

for which the adjoint is

$$N(v) \equiv e_{2m}(x,\lambda)\frac{d^{2m}v}{dx^{2m}} + \lambda^2 e_{2m-2}(x,\lambda)\frac{d^{2m-2}v}{dx^{2m-2}} + \cdots$$

$$+ \lambda^{2m-2} e_2(x,\lambda)\frac{d^2 v}{dx^2} + \lambda^{2m} e_0(x,\lambda)v = 0. \tag{14}$$

It is clear that in this discussion the coefficients p_i, $i = 2, 4, \cdots, 2m$, of the auxiliary equation $P(y) = 0$ will be chosen identical respectively with e_i where i has the values $2, 4, \cdots, 2m$, and ~~if~~ for $i = 1, 3, \cdots, 2m-1$ the functions p_i will be identically zero. The functions $e_{2m}, e_{2m-2}, \cdots, e_2, e_0$ are to be taken as functions of x and of λ having the same properties as those specified for the corresponding coefficients of $L(u) = 0$.

[illegible]

The following theorem may be stated as a special case of theorem I.

Theorem Ia. *If the parameter λ, in the equation $E(u) = 0$, is fixed and satisfies the relations,*

$$\lambda \geq \bar{\lambda}\ ,\ \lambda \geq \frac{s\pi}{b-a}\ ,$$

and if for every x in the interval $a \leq x \leq b$

$$e_0(x,\lambda) \geq A_0 + A_2\tan^2\lambda(x-c) + \cdots + A_{2m-2}\tan^{2m-2}\lambda(x-c),\ c = a - \pi/2\lambda\ ,$$

the equality sign not holding throughout (a,b), then any solution $u(x,\lambda)$ of $E(u) = 0$ vanishes at least once in the interior of each sub-interval of (a,b) of length π/λ measuring from the point a and hence has at least s zeros in (a,b).

Let us suppose for the moment that the coefficients $e_0, e_2, \cdots, e_{2m}$ of $E(u) = 0$ are not functions of x and λ but of x alone. Then the coefficients $A_0, A_2, \cdots, A_{2m-2}$ are functions of x alone; and, we see that if the functions $A_2, A_4, \cdots, A_{2m-2}$ are either negative or zero for every x in the interval (a,b) and if $e_0(x) \geq A_0$, $(e_0(x) \not\equiv A_0)$, we may conclude that as λ increases to infinity the number of zeros which we are assured exist in the given interval increases indefinitely.

V. Second General Theorem of Oscillation.

The method of §III affords a means of deriving other general theorems of oscillation than the one of that section; for we may choose different particular solutions of a suitable comparison equation $P(y) = 0$ and hence obtain different values of p_0 to be used in the inequality $p_0 - m_0 \leq 0$. In place of the cosine function which was used in the section just referred to, let us select as the particular solution of a comparison equation $P(y) = 0$, suitably chosen, $y_2 = cn^{n-1}\lambda(x-d)$, where $cn\, x$ is one of the Jacobi elliptic functions and where d $(= a - K/\lambda)$ is a constant chosen so that y_2 vanishes at the point $\underline{a}$ of the interval (a,b). We shall use only real values of the argument x and hence for our purpose $sn\, x$, $dn\, x$ and $cn\, x$ have no singularities for finite values of x; $sn\, x$ has simple zeros at points congruent modulo $4K$ with $x = 0$ and $x = 2K$ where K is the quarter period of these functions, $dn\, x$ has no (real) zeros and $cn\, x$ has simple zeros at points congruent modulo $4K$ with $x = K$ and $x = 3K$. We will again choose the functions $p_n, p_{n-1}, \cdots, p_1$ identical respectively with $m_n, m_{n-1}, \cdots, m_1$ so that we have $D(y_2) \equiv \lambda^n (p_0 - m_0)\, y_2$. From previous considerations it is clear that m_0 is expressible linearly in terms of the functions l_0 and $l_i^{(i)}$, $i = 1, 2, \cdots, n$, and it remains for us to compute the value of p_0 in terms of the coefficients of $L(u)$ and their derivatives.

The substitution of the particular solution y_2 for y in $P(y) = 0$ may be easily carried out if we first obtain the successive derivatives of y_2. The amount of computation may be reduced if we place

$$y_2^{(\lambda)} = w_\lambda y_2,$$

from which we have the ~~partial~~ difference and differential recurrence relation

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$w_{r+1} = w_r' - \lambda(n-1)\frac{\operatorname{sn}\lambda(x-d)\ \operatorname{dn}\lambda(x-d)}{\operatorname{cn}\lambda(x-d)}\, w_r ,$$

where r is the recurrence variable. We have

$$y_2 = \operatorname{cn}^{n-1}\lambda(x-d) ,$$

$$y_2' = \left[-(n-1)\frac{\operatorname{sn}\lambda(x-d)\operatorname{dn}\lambda(x-d)}{\operatorname{cn}\lambda(x-d)}\right]\lambda\, y_2 ,$$

$$y_2'' = \left[(n-1)(n-2)\frac{\operatorname{sn}^2\lambda(x-d)\operatorname{dn}^2\lambda(x-d)}{\operatorname{cn}^2\lambda(x-d)} + (n-1)\left\{k^2\operatorname{sn}^2\lambda(x-d) - \operatorname{dn}^2\lambda(x-d)\right\}\right]\lambda^2 y_2.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .$$

Multiplying y_2 and its first n derivatives in order by $\lambda^n p_0$, $\lambda^{n-1}p_1, \cdots$, λp_{n-1}, p_n and adding, we obtain for the left hand member $P(y_2)$; and this has the value zero. Solving the resulting equation for p_0 and arranging its value according to the powers of the function $\operatorname{sn}\lambda(x-d)\operatorname{dn}\lambda(x-d)/\operatorname{cn}\lambda(x-d)$, we have

$$p_0 = B_0 + B_1\frac{\operatorname{sn}\lambda(x-d)\ \operatorname{dn}\lambda(x-d)}{\operatorname{cn}\lambda(x-d)} + \cdots + B_{n-1}\frac{\operatorname{sn}^{n-1}\lambda(x-d)\operatorname{dn}^{n-1}\lambda(x-d)}{\operatorname{cn}^{n-1}\lambda(x-d)} ,$$

where the functions B_i, $i = 0, 1, 2, \cdots, n-1$, are expressible linearly in terms of the functions p_k, $k = 1, 2, \cdots, n$, and contain only those p's for which k is even or odd according as i is even or odd respectively and for which $k \geq i$. Moreover, in B_i, the coefficient of p_k, $k = i+2, i+4, \cdots$, are not constants but are functions of x and λ which are finite for x in (a,b) and for a fixed value of λ in the range $\lambda \geq \bar{\lambda}$ while the coefficient of p_k for $k = i$, is a constant different from zero. In §VII we shall give an example to exhibit the importance of the fact that the coefficients of the p's are not all constants. As in the preceding section the p's are expressible linearly in terms of the l's and their derivatives and hence the functions B_i, $i = 0, 1, 2, \cdots, n-1$ are determinate functions of the coefficients of the given equation.

Since $D(y_2) \equiv \lambda^n(p_0 - m_0)y_2$ where $p_0 - m_0$ is a determinate function of the coefficients of L(u), linear in l_0 and not involving its derivatives of any order, if we apply the theorem of §II for this case in a manner precisely analogous to that used in the proof of theorem I, we can establish the following theorem which will be called the Second General Theorem of Oscillation.

Theorem II. *If the parameter λ, in the equation $L(u) = 0$, is fixed and satisfies the relations,*

$$\lambda \geq \bar{\lambda}\ ,\ \lambda \geq \frac{2Ks}{b - a}\ ,$$

and if for every x in the interval $a \leq x \leq b$

$$l_0(x,\lambda) \geq B_0 + B_1\frac{\mathrm{sn}\lambda(x-d)\,\mathrm{dn}\lambda(x-d)}{\mathrm{cn}\lambda(x-d)} + \cdots + B_{n-1}\frac{\mathrm{sn}^{n-1}\lambda(x-d)\,\mathrm{dn}^{n-1}\lambda(x-d)}{\mathrm{cn}^{n-1}\lambda(x-d)}$$

$$+ \frac{l_1'}{\lambda} - \frac{l_2''}{\lambda^2} + \cdots + (-1)^{n+1}\frac{l_n^{(n)}}{\lambda^n}\ ,\quad d = a - \frac{K}{\lambda}\ ,$$

the sign of equality not holding throughout (a,b), *then any solution* $u(x,\lambda)$ *of* $L(u) = 0$ *vanishes at least once in the interior of each sub-interval of* (a,b) *of length* $2K/\lambda$ *measuring from the point a and hence has at least s zeros in* (a,b):

1) *when n is even;*

2) *when n is odd, provided in this case that* $u(x,\lambda)$ *is positive at some point in each of the intervals* $(\mu_0,\mu_1), (\mu_1,\mu_2), \cdots, (\mu_{\lambda-1},\mu_\lambda)$, *and that*

$$l_n(\mu_0,\lambda)y_2^{(n-1)}(\mu_0,\lambda)u(\mu_0,\lambda) \leq l_n(\mu_1,\lambda)y_2^{(n-1)}(\mu_1,\lambda)u(\mu_1,\lambda) \leq$$

$$\cdots \leq l_n(\mu_\lambda,\lambda)y_2^{(n-1)}(\mu_\lambda,\lambda)u(\mu_\lambda,\lambda),$$

where $\mu_0 = a, \mu_1, \cdots, \mu_\lambda$ *are the consecutive zeros of* y_2 *in* (a,b).

[illegible]

[illegible]
[illegible]
[illegible]
[illegible]
[illegible]

[illegible]
[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

$$[illegible]$$

[illegible]
[illegible]
[illegible]
[illegible]

[illegible]

[illegible]
[illegible]
[illegible]

$$[illegible]$$

$$[illegible]$$

[illegible]

In order that the theorem may be more easily applied to a particular equation it is evident from considerations analogous to those of the previous section that it will be convenient to confine attention to an equation of even order which is such that the coefficients of the odd powers of $\operatorname{sn}\lambda(x-d)\operatorname{dn}\lambda(x-d)/\operatorname{cn}\lambda(x-d)$ identically zero. This restriction again makes it necessary that the p's with odd subscripts are identically zero for the coefficient of p_i in B_i, $i = 1,3,5,\cdots$, is a constant different different from zero. Since the functions p_i are chosen identical with the functions m_i, $i = 1,3,5,\cdots$, the equation again reduces to $E(u) = 0$ as defined in (13) and which has as its adjoint $N(v) = 0$ as given by (14).

The following theorem is a special case of theorem II and forms an analogue of theorem Ia.

Theorem IIa. *If the parameter* λ, *in the equation* $E(u) = 0$, *is fixed and satisfies the relations,*

$$\lambda \geq \bar{\lambda}\ ,\ \lambda \geq \frac{2Ks}{b-a}\ ,$$

and if for every x *in the interval* $a \leq x \leq b$

$$e_0(x,\lambda) \geq B_0 + B_2\frac{\operatorname{sn}^2\lambda(x-d)\operatorname{dn}^2\lambda(x-d)}{\operatorname{cn}^2\lambda(x-d)} + \cdots + B_{2m-2}\frac{\operatorname{sn}^{2m-2}\lambda(x-d)\operatorname{dn}^{2m-2}\lambda(x-d)}{\operatorname{cn}^{2m-2}\lambda(x-d)}\ ,$$

$d = a - K/\lambda$, *the sign of equality not holding throughout* (a,b), *then any solution* $u(x,\lambda)$ *of* $E(u) = 0$ *vanishes at least once in the interior of each sub-interval of* (a,b) *of length* $2K/\lambda$ *measuring from the point* a *and hence has at least* s *zeros in* (a,b).

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

[illegible]

VI. Third General Theorem of Oscillation.

If, in §III we had used the sine function instead of the cosine function as a particular solution of a comparison equation $P(y) = 0$ we would have obtained no results different from those derived by the use of the cosine, for the sine curve is transformable into the cosine curve by translating it a distance $\pi/2$ parallel to the x-axis. Such a relation does not exist between the elliptic functions $cn\, x$ and $sn\, x$ for if $y = cn\, x$ is translated a distance $-K$ parallel to the x-axis we have $y = cn(x-K) = k' sn\, x / dn\, x$ where $k' = \sqrt{1-k^2}$; hence if we use a power of $sn\, x$ as a solution of a comparison equation $P(y) = 0$ we will get results which are different from those obtained by using $cn^{n-1}\lambda(x-d)$ as such a solution.

We will choose as a particular solution of a suitable comparison equation $P(y) = 0$ the function $y_3 = sn^{n-1}\lambda(x-a)$ where the constant $\underline{a}$ is the affix of the leftmost point of the interval $a \leq x \leq b$. We will not carry out the details of the preceding section for the new function y_3 but will note that no new assumptions are made on the functions involved and that nothing different is obtained until we take the successive derivatives of y_3. We have

$$y_3 = sn^{n-1}\lambda(x-a),$$

$$y_3' = \left[(n-1)\frac{cn\,\lambda(x-a)\,dn\,\lambda(x-a)}{sn\,\lambda(x-a)}\right]\lambda y_3 \; ,$$

$$y_3'' = \left[(n-1)(n-2)\frac{cn^2\lambda(x-a)\,dn^2\lambda(x-a)}{sn^2\lambda(x-a)} - (n-1)\left\{dn^2\lambda(x-a) + k^2 sn^2\lambda(x-a)\right\}\right]\lambda^2 y_3 \; ,$$

. . . .

It will be convenient again to place

$$y_2^{(r)} = w_r y_2 ;$$

from this equation we obtain the partial difference and differential recurrence relation

$$w_{r+1} = w_r' + \lambda(n-1)\frac{\mathrm{cn}\,\lambda(x-a)\,\mathrm{dn}\,\lambda(x-a)}{\mathrm{sn}\,\lambda(x-a)}\, w_r ,$$

r being the recurrence variable. This relation will be found useful in making the computations for any particular case. The value of p_0, found in precisely the same way as in the previous section, is

$$p_0 = C_0 + C_1\frac{\mathrm{cn}\,\lambda(x-a)\,\mathrm{dn}\,\lambda(x-a)}{\mathrm{sn}\,\lambda(x-a)} + \cdots + C_{n-1}\frac{\mathrm{cn}^{n-1}\lambda(x-a)\,\mathrm{dn}^{n-1}\lambda(x-a)}{\mathrm{sn}^{n-1}\lambda(x-a)} ,$$

where the coefficients C_i are functions analogous to the coefficients B_i of the previous section, $i = 0, 1, \cdots, n-1$, the only difference being that in these functions the coefficients of p_k, $k = i, i+2, \cdots$, are not all the same for the two cases.

We are now justified in stating the Third General Theorem of Oscillation, the proof for which follows exactly the proof of theorem I.

Theorem III. _If the parameter_ λ, _in the equation_ $L(u) = 0$, _is fixed and satisfies the relations,_

$$\lambda \geq \bar{\lambda} \quad , \lambda \geq \frac{2Ks}{b - a} ,$$

and if for every x _in the interval_ $a \leqq x \leqq b$

$$l_0(x,\lambda) \geq C_0 + C_1\frac{\mathrm{cn}\,\lambda(x-a)\,\mathrm{dn}\,\lambda(x-a)}{\mathrm{sn}\,\lambda(x-a)} + \cdots + C_{n-1}\frac{\mathrm{cn}^{n-1}\lambda(x-a)\,\mathrm{dn}^{n-1}\lambda(x-a)}{\mathrm{sn}^{n-1}\lambda(x-a)}$$

$$+ \frac{l_1'}{\lambda} - \frac{l_2''}{\lambda^2} + \cdots + (-1)^{n+1}\frac{l_n^{(n)}}{\lambda^n} ,$$

the sign of equality not holding throughout (a,b), _then any solution_ $u(x,\lambda)$

[illegible]

of $L(u) = 0$ vanishes at least once in the interior of each sub-interval of (a,b) of length $2K/\lambda$ measuring from the point a and hence has at least s zeros in (a,b) :

1) when n is even;

2) when n is odd, provided in this case that $u(x,\lambda)$ is positive at some point in each of the intervals $(\mu_0,\mu_1), (\mu_1,\mu_2), \cdots, (\mu_{s-1},\mu_s)$, and that

$$l_n(\mu_0,\lambda)v_2^{(n-1)}(\mu_0,\lambda)u(\mu_0,\lambda) \leq l_n(\mu_1,\lambda)v_2^{(n-1)}(\mu_1,\lambda)u(\mu_1,\lambda) \leq \cdots \leq l_n(\mu_s,\lambda)v_2^{(n-1)}(\mu_s,\lambda)u(\mu_s,\lambda) .$$

where $\mu_0 = a, \mu_1, \cdots, \mu_s$ are the consecutives zeros of v_2 in (a,b).

It is clear from the discussion just preceding the statement of theorem IIa that, in this case also, the most convenient application of theorem III will be that to the equation $E(u) = 0$ instead of $L(u) = 0$. A special case of theorem III gives the following theorem as a second analogue of theorem Ia.

Theorem IIIa. If the parameter λ, in the equation $E(u) = 0$, is fixed and satisfies the relations,

$$\lambda \geq \bar{\lambda} \quad , \lambda \geq \frac{2Ks}{b-a} ,$$

and if for every x in the interval $a \leq x \leq b$

$$e_0(x,\lambda) \geq C_0 + C_2\frac{cn^2\lambda(x-a)dn^2\lambda(x-a)}{sn^2\lambda(x-a)} + \cdots + C_{2m-2}\frac{cn^{2m-2}\lambda(x-a)dn^{2m-2}\lambda(x-a)}{sn^{2m-2}\lambda(x-a)} ,$$

the sign of equality not holding throughout (a,b), then any solution $u(x,\lambda)$ of $E(u) = 0$ vanishes at least once in the interior of each sub-interval of (a,b) of length $2K/\lambda$ measuring from the point a and hence has at least s zeros in (a,b).

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

[illegible]

VII. Application to Equations not Involving a Parameter.

For the application of the above theorems let us consider the equation

$$\bar{E}(u) \equiv \frac{d^{2m}}{dx^{2m}}\left[E_{2m}(x)u\right] + \frac{d^{2m-2}}{dx^{2m-2}}\left[E_{2m-2}(x)u\right] + \cdots + \frac{d^2}{dx^2}\left[E_2(x)u\right] + E_0(x)u = 0,$$

in which the functions $E_0, E_1, \cdots, E_{2m}$ and their derivatives of as high order as their subscripts are real-valued, single-valued and continuous functions of the real variable x for x in the interval $a \leq x \leq b$. We will suppose that the coefficient $E_{2m}(x)$ of the derivative of highest order is of one sign, say positive, throughout (a,b). An equation of the above type may be rewritten in the form

$$\bar{E}(u) \equiv \frac{d^{2m}}{dx^{2m}}\left[E_{2m}(x)u\right] + \lambda^2\frac{d^{2m-2}}{dx^{2m-2}}\left[\frac{E_{2m-2}(x)}{\lambda^2}u\right] + \cdots + \lambda^{2m-2}\frac{d^2}{dx^2}\left[\frac{E_2(x)}{\lambda^{2m-2}}u\right] + \lambda^{2m}\frac{E_0(x)}{\lambda^{2m}}u = 0.$$

Applying theorem Ia to this equation we have in place of the condition on e_0

$$\frac{E_0(x)}{\lambda^{2m}} \geq A_0 + A_2\tan^2\lambda(x-c) + \cdots + A_{2m-2}\tan^{2m-2}\lambda(x-c),$$

where, in the coefficients $A_0, A_2, \cdots, A_{2m-2}$ the functions p_{2m-2i} have the special form E_{2m-2i}/λ^{2i}, $i = 0, 1, \cdots, m-1$.

Except in very special cases we may consider that the functions $A_2, A_4, \cdots, A_{2m-2}$ (when not identically zero) do not have zeros to balance the poles of $\tan\lambda(x-c)$ as λ varies; hence this inequality, which, if satisfied by $E_0(x)/\lambda^{2m}$, assures us of the existence of zeros of a solution of the given equation, may be satisfied in general only if the coefficient of each power of $\tan\lambda(x-c)$ is less than or equal to zero for certain values of λ and for x in certain sub-intervals of the given interval. Hence it is nat-

[illegible]

ural to suppose that this equation is so restricted that

$$A_{2i} \leq 0, \; i = 1, 2, \cdots, m-1,$$

throughout (a,b). The maximum value of the sum of all the terms, in the second member of the inequality on $E_0(x)/\lambda^{2m}$, involving the tangent function, is zero; if these terms are neglected we have a new condition on $E_0(x)/\lambda^{2m}$ which is

$$A_0 \leq \frac{E_0(x)}{\lambda^{2m}} \; .$$

Suppose that in the above inequalities and in the function $y_1 = \cos^{2m-1}\lambda(x-c)$, we substitute π/l for λ. The functions $A_0, A_2, \cdots, A_{2m-2}$ after this substitution will be denoted by $\bar{A}_0, \bar{A}_2, \cdots, \bar{A}_{2m-2}$, respectively. Now if the constant c is so chosen that y_1 has a zero at one end of any given sub-interval of (a,b) of length l, the next zero will be at the other end of this sub-interval, for the half period of $\cos\lambda(x-c)$ is π/λ or l. We note here that, if the above inequalities hold, then the condition imposed upon $D(y_1)$ in theorem Ia is satisfied; moreover any change of the constant c does not in any way affect these inequalities. Hence these inequalities may be looked upon as conditions on l; the smallest l which will satisfy these conditions for every x in this subinterval will give the most information concerning the lengths of sub-intervals of (a,b) in which we can be sure of finding at least one zero of any solution of the given differential equation. The results which thus emerge may be stated in the following theorem.

Theorem IV. *If, in a given interval* $a \leq x \leq b$, *there is a sub-interval of length l (the smallest l being chosen which will satisfy the following inequalities), such that for every x in this sub-interval*

$$\bar{A}_{2i} \leq 0, \quad i = 1, 2, \cdots, m-1,$$

$$\bar{A}_0 \leq E_0(x)\frac{l^{2m}}{\pi^{2m}},$$

not all the equality signs holding throughout the sub-interval l, then any solution u(x) of the differential equation $\bar{E}(u) = 0$ has at least one zero in the interior of this sub-interval.

Since we can equally well apply theorem IIa or IIIa to the equation $\bar{E}(u) = 0$ two other theorems might be stated, but the only change would be that the $\bar{A}$'s in the inequalities would be replaced by analogous $\bar{B}$'s or $\bar{C}$'s corresponding to the B's or C's of §5 or §6, respectively, according as we applied theorem IIa or IIIa, and that in either case π, the half period of sin x and cos x would give place to 2K the half period of the Jacobi elliptic functions cn x and sn x.

Let us apply these results to the special cases in which m = 1 or m = 2.

If m = 1 we may, without loss of generality, take $E_2(x) = 1$. Then in place of the inequalities of the above theorem, we have

$$l^2 E_0(x) \geq \pi^2 .$$

This inequality can be satisfied only if $E_0(x)$ is positive throughout the sub-interval of length l; and when the inequality is true, the other hypotheses of theorem IV being satisfied, we are assured of the existence of at least one zero of a solution u(x) of the given second order equation $u'' + E_0(x)u = 0$ in this sub-interval. Thus it is clear that our theorem IV is a generalization of one of the well-known Sturmian* results for a second

* M. Bôcher, Bulletin of the American Mathematical Society, (2) vol. 4 (1898), pp. 295-313.

order equation in binomial form.

It should be pointed out here that if we had applied theorem IIa to the equation $\bar{E}(u) = 0$ as we applied theorem Ia and had then specialized the results to the second order equation in binomial form, we would have for the inequality on $E_0(x)$,

$$l^2 E_0(x) \geq \left[1 - 2k^2 \mathrm{sn}^2 \frac{2K}{l}(x-d)\right](2K)^2 .$$

Since $2k^2 \mathrm{sn}^2 \frac{2K}{l}(x-d)$ is always positive or zero and the maximum value is $2k^2 < 2$ it is clear that $E_0(x)$ may be less than π^2/l^2 for x in certain parts of the sub-interval l and still satisfy this relation where we have the same l for this case that we had in $l^2 E_0(x) \geq \pi^2$. Using both of these inequalities, the range of values for $E_0(x)$ which make theorem IV applicable, is thus extended beyond the limit π^2/l^2 of the foregoing result. This same extension may be also obtained from the well-known comparison theorem for two equations where in particular cn x is a solution of one of the equations.

The inequalities of theorem IV for the case of the fourth order equation obtained by letting m = 2 are

$$l^2 E_2(x) - 10\pi^2 E_4(x) \geq 0,$$

$$l^4 E_0(x) \geq 3\pi^2 l^2 E_2(x) - 21\pi^4 E_4(x) .$$

We are assuming that $E_4(x)$ is positive; hence if $E_2(x)$ and $E_0(x)$ are both positive, the constant l can be chosen large enough to make the inequalities true provided for all values of x considered the functions are bounded away from zero and infinity. Considering l as the length of a sub-interval of the given interval, the conditions imposed by the theorem are evident. For convenience in stating certain results we will suppose that instead of the

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

interval (a,b) we are considering the interval $a \leq x < +\infty$ which includes any sub-interval l however large l may be. Then if the functions $E_i(x)$ and their derivatives up to order i inclusive, i = 0,2,4, are real-valued, single-valued and continuous functions of x in $a \leq x < +\infty$ and are bounded away from zero and infinity in this interval, then a sub-interval l of $a \leq x < +\infty$, measured from any fixed point of the given interval, exists which contains at least one zero of every solution of a fourth order equation in the form of $\bar{E}(u) = 0$.

For the general case of order 2m the sign of p_{2m-2i} or for this case of $l^{2i}E_{2m-2i}(x)/\pi^{2i}$ in $\bar{A}_{2m-2i}$, $i = 1,2,\cdots,m-1$, is negative. l^{2i} is the highest power of l entering in $\bar{A}_{2m-2i}$. If for every x in the interval $a \leq x < +\infty$ the functions $E_{2m-2k}(x)$ and their derivatives up to as high order as 2m-2k are real-valued, single-valued and continuous and have a finite upper bound and if $E_{2m-2k}(x) > Mx^{-2+\epsilon}$, where M and ϵ are fixed positive constants, then l may always be chosen large enough to make the functions $\bar{A}_{2m-2i}$ negative or zero and thus to satisfy the first m-1 inequalities of theorem IV, since for large values of l, $l^{2i}E_{2m-2i}$ is of higher order of magnitude than any other terms of $\bar{A}_{2m-2i}$; moreover $l^{2m}E_0(x)$ is of higher degree in l than any terms of A_0 and thus the last inequality of the theorem may be satisfied. Hence if we assume that in the equation $\bar{E}(u) = 0$ the functions $E_0(x)$, $E_2(x)$, $\cdots$, $E_{2m}(x)$ and their derivatives of order as high as their subscripts are real-valued, single-valued and continuous functions of x for x in the interval $a \leq x < +\infty$, we can state the following result as a corollary of theorem IV.

Corollary. _If for every x in the interval_ $a \leq x < +\infty$ _the coefficients_ $E_{2m-2k}(x)$ _and their derivatives up to order as high as 2m-2k inclusive_, $k = 0,1,2,\cdots,m$, _are real-valued, single-valued and continuous and_

have a finite upper bound and if $E_{2m-2k}(x) > Mx^{-2+\epsilon}$ *where* M *and* ϵ *are fixed positive constants, then it is always possible to find a sub-interval of length* 1, *measured from any fixed point of the given interval, such that every solution of the equation* $\bar{E}(u) = 0$ *has at least one zero in the interior of this sub-interval.**

The result stated by this corollary for a second order equation in binomial form may also be obtained as an immediate consequence of the classic Sturmian theorem referred to above. Hence, we have theorems which, although retaining the simplicity of form of certain of the Sturmian results, are nevertheless applicable to a class of equations of general even order. The general oscillation problem for this class of equations is still unsolved; for our theorems state only sufficient conditions that every solution of a given equation vanishes in a suitably determined interval.

....

* It is evident that corresponding results may be stated for the interval $-\infty < x \leq b$.

[illegible]

[illegible]

[illegible]

ACKNOWLEDGMENT.

The writer wishes to express his gratitude for the kindly aid and helpful suggestions of Professor R. D. Carmichael who supervised the preparation of this paper.

VITA.

Merlin Grant Smith was born on a farm near Delta, Ohio, August26, 1894. His work in the grades was divided into short periods taken at schools in various parts of the state. After spending the first two years of high school at Pomeroy, Ohio, the writer moved with his parents to Youngstown, Ohio, where he entered Rayen High School from which, two years later, in 1911, he was graduated.

The following four years were spent in Greenville College, Greenville, Illinois, by which , in 1915, the writer was granted the degree of Bachelor of Arts. Besides being valedictorian of his class he was awarded a scholarship by the University of Illinois. This scholarship was held throughout the year 1915-1916, and in June 1916 he was granted the degree of Master of Arts by the University of Illinois. The years 1916-1917 and 1917-1918 were spent as a fellow in mathematics at the University of Illinois.

While in attendance at the University of Illinois the writer was elected to active membership in the Illinois chapter of Sigma Xi.